A framework of high precision eigenvalue estimation for self-adjoint elliptic differential operator

Xuefeng LIU

Research Institue for Science and Engineering
Waseda Univeristy, Japan

Cooperated with
Prof. S. OISHI, Waseda University, Japan
Prof. M. Plum, Karlsruhe University, Germany

September 23, 2012

Outline

1. Motivation and background of eigenvalue evaluation
2. The framework for high precision eigenvalue bounds
3. The eigenvalue bounds for base problem
4. Computation examples.

1. Motivation and background of eigenvalue evaluation

Motivation of this research

- Explicit error estimation of approximate solution u_h obtained by adopting finite element method(FEM):

$$\|u - u_h\|_{H_0^1(\Omega)} \le Ch\|f\|_{L_2} \quad \text{C: the interpolation error constant}$$

- Bounds of the constant in embedding theorem: (Reduced to eigenvalue problem of $-\Delta$)

$$\|u\|_{L_2(\Omega)} \le C|u|_{H_0^1(\Omega)} \text{ for } u \in H_0^1(\Omega)$$

- Spectrum evaluation of differential operator needed in investigating the solution existence.

$$Lu := -\Delta u + cu, \quad Lu = f \in \Omega, \text{ asso. with b.d.c.}$$

- e.t.c.

Objective

On 2D polygonal domain Ω, consider the eigenvalue of elliptic operator L:

$$Lu := -\text{div}(A\nabla u) + cu, \quad Lu = \lambda u \quad \text{associated with b.d.c.}$$

where $A \in C^1(\Omega)$, $c \in L_\infty(\Omega)$.

Classical results :

- There are many numerical schemes(FEM, FDM, FVM) along with theoretical analysis to give **approximate result**;
- It is difficult to give concrete **lower bound** for the eigenvalues;
- Reference book, e.g., I. Babuska, J. Osborn, *Eigenvalue Problems, Finite Element Methods (Part 1)*., Elsevier Science Publishers B.V., 1991

Recent results:

- Nakao's method: based on invariant point theorem;
- Plum's method: based on homotopy theorem.

2. The framework for high precision eigenvalue bounds

Challenges in desiring high precision bounds

Katou's bound [Katou, 1949]

Consider the eigenvalue problem of $-\Delta$. Let $\tilde{u} \in D(\Delta)$ be approximate eigenvector, and $\tilde{\lambda} := \|\nabla\tilde{u}\|_\Omega^2/\|\tilde{u}\|^2$ and $\sigma := \|-\Delta\tilde{u} - \tilde{\lambda}\tilde{u}\|/\|\tilde{u}\|_\Omega$ (*Weinstein's bound*).
Suppose that μ and ν satisfy, for certain n,

$$\lambda_{n-1} \le \mu < \tilde{\lambda} < \nu \le \lambda_{n+1}. \qquad \text{[a priori estimation]}$$

Thus,

$$\tilde{\lambda} - \frac{\sigma^2}{\nu - \tilde{\lambda}} \le \lambda_n \le \tilde{\lambda} + \frac{\sigma^2}{\tilde{\lambda} - \mu} \qquad \text{[precision: } O(\sigma^2) \text{]}$$

- A priori estimation of index is needed;
- Well-constructed vector $\hat{u}$ can provide high precision bounds;

Lehmann-Goerisch's theorem can be regarded as extended version of Katou's bound, which can easily deal with clustered eigenvalues.

Lehmann-Goerisch's theorem

Lehmann's theorem [Lehmann, 1963]

Pre-requisition of Lehmann's method:

- A priori estimation such that for certain ν satisfying $\lambda_m < \nu \leq \lambda_{m+1}$.
- Approximate eigenvector $\{v_1, \cdots, v_m\}$ in $D(\Delta)$.

Estimation from Lehmann's method: Introduce an auxiliary $m \times m$ eigen-problem $B_1 x = \mu B_2 x$ based on ν and $\{v_i\}_{i=1,m}$, which has the eigenvalues,

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$$

Then,

$$\lambda_{m+1-k} \geq \nu + \frac{1}{\mu_k}, \quad (k = 1, \cdots, m)$$

Goerisch's result can weaken the condition for v_i, i.e., $v_i \in D(\Delta)$.

Homotopy method

M. Plum applied the homotopy theorem to give the a priori estimation needed by Lehmann-Goerisch's theorem ,

$$\lambda_k < \nu \leq \lambda_{k+1} \ .$$

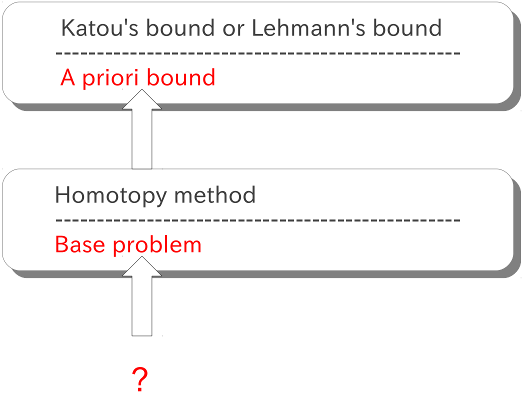
Homotopy method [Plum, 1991]

Base problem : operator A_0	$\rightarrow$	Objective problem: operator A_1
$-\Delta u = \lambda u$	$\rightarrow$	$-\text{div}(A\nabla u) + cu = \lambda u$
Explicit spectrum: $\lambda_1^0 \leq \lambda_2^0 \leq \cdots$		Spectrum to estimate: $\lambda_1^1 \leq \lambda_2^1 \leq \cdots$

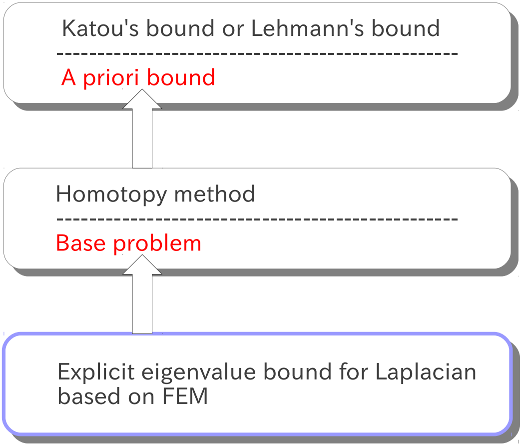
An intermediate operator is introduced, $A_s := (1 - s)A_0 + sA_1$
The eigenvalues of A_s, denoted by $\{\lambda_k^s\}$, are supposed to satisfy

$$\lambda_k^0 \leq \lambda_k^{s_1} \leq \lambda_k^{s_2} \leq \lambda_k^1, \quad 0 \leq s_1 \leq s_2 \leq 1.$$

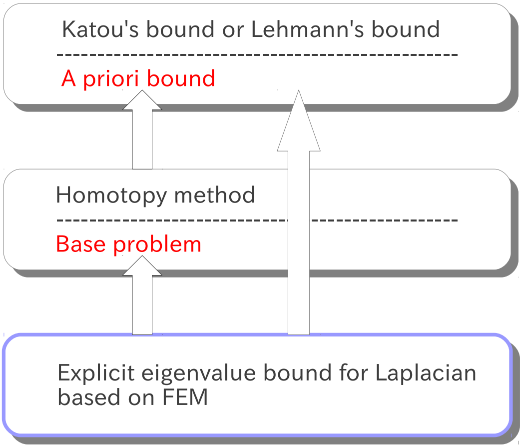
Challenges in desiring high precision bounds



Challenges in desiring high precision bounds



Challenges in desiring high precision bounds



3. Explicit eigen-bound for Laplacian based on FEM

Eigenvalue bounds for Laplacian

Assumption: Ω to be 2D bounded polygonal domain.

Eigenvalue problem of Laplacian

$$-\Delta u = \lambda u \text{ in } \Omega, u = 0 \text{ on } \partial\Omega\,.$$

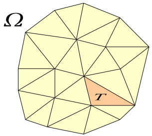
Weak formulation: Find $u \in H_0^1(\Omega)$ and $\lambda \in R$ s.t.

$$(\nabla u, \nabla v)_\Omega = \lambda (u, v)_\Omega, \quad \forall v \in V \cap H_0^1(\Omega)\,.$$

- Eigen-pairs: $\{\lambda_k, u_k\}_{k=1,\infty}$, $(\lambda_k \le \lambda_{k+1})$.

Eigenvalue bounds for Laplacian

Finite element method is adoptted.



Triangulation of domain: $\mathcal{T}^h$

Let V_h be the piece-wise linear continuous finite element space over $\mathcal{T}^h$.

Approximate eigenvalues by FEM: Find $u_h \in V_h \cap H_0^1(\Omega)$ and $\lambda_h \in R$ s.t.

$$(\nabla u_h, \nabla v_h)_\Omega = \lambda_h (u_h, v_h)_\Omega, \quad \forall v_h \in V_h \cap H_0^1(\Omega) .$$

- Eigen-pairs: $\{\lambda_{k,h}, u_{k,h}\}_{k=1,n}$, $(\lambda_{k,h} \le \lambda_{k+1,h})$.
- Theoretically, we know $|\lambda_{k,h} - \lambda_k| = O(h^2)$.

Eigenvalue bounds for Laplacian

Theorem [Liu-Oishi 2010] Let $\{\lambda_k, u_k\}$ and $\{\lambda_{k,h}, u_{k,h}\}$ defined as before. By applying Max-Min principle, we obtain bounds for λ_k:

$$\frac{\lambda_{k,h}}{1 + M(h)^2 \lambda_{k,h}} \le \lambda_k \le \lambda_{k,h} \ . \tag{1}$$

- The constant $M(h)$ is dependent on the mesh triangulation;
- For convex domain, $M(h)$ is easy to obtain. For **non-convex domain**, it is very difficult to give concrete value.

Explicit estimation of $M(h)$

Quantitative a priori error estimate on convex or non-convex domain:

Theorem [Liu 2012] Given $f \in L_2(\Omega)$, let $u \in H_0^1(\Omega)$ and $u_h \in V_h$ be the solutions of variational problems below, respectively,

$$(\nabla u, \nabla v) = (f, v) \quad \text{for } v \in H_0^1(\Omega)\,, \quad (\nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for } v_h \in V_h(\Omega)\,.$$

Let $M(h) := \sqrt{C_{0,h}^2 + \kappa_h^2}$, we have error estimates as below,

$$\|\nabla(u - u_h)\|_{L_2} \le M(h)\|f\|_{L_2}, \quad \|u - u_h\|_{L_2} \le M(h)^2\|f\|_{L_2}$$

where κ_h is defined by

$$\kappa_h := \sup_{f_h \in M^h \setminus 0} \inf_{p_h \in W^h_{f_h}} \inf_{u_h \in S^h} \frac{\|p_h - \nabla u_h\|}{\|f_h\|}$$

FEM spaces: W^h, M^h, W_{f_h}

Over **triangular** mesh $\mathcal{T}^h$.

- V_h: piece-wise linear continuous finite element space.
- Raviart-Thomas mixed FEM space W^h:

$$W^h := \{ p_h \in H^1(div, \Omega) | p_h|_k = (a_K + c_K x, b_K + c_K y) \text{ on each element } K \in \mathcal{T}^h \} .$$

- Piecewise constant function space M^h:

$$M^h = \{ \text{piecewise constant on the triangulation } T^h \}$$

- Subspace of W^h related to $f_h \in M^h$:

$$W^h_{f_h} := \{ p_h \in W^h \mid \text{div } p_h + f_h = 0 \text{ on each } K \in \mathcal{T}^h \} .$$

- div $(W^h) = M^h$.

4. Numerical examples

4. Numerical Examples

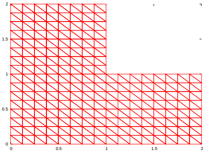
Computing environment

For purpose of verified computation:

- Software/Library:
  - Boost Interval C++ library, INTLAB interval toolbox of Matlab.
- Algorithm for verified evaluation of matrix eigenvalues: improved version of
  - H. Behnke. The calculation of guaranteed bounds for eigenvalues using complementary variational principles. Springer-Verlag, 47(1):11-27, 1991.

Example I: L-shape domain

L-shape domain $\Omega : (0,2) \times (0,2) \setminus [1,2] \times [1,2]$.
A priori bound: $\lambda_5 < \nu = 39 < \lambda_6$
Degree of polynomial bases: $N = 5, 10$

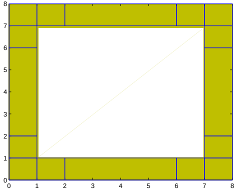


λ_i	Fox,1967	He & Yuan,2009	Liu-Oishi (N=6)	Liu-Oishi (N=10)
1	9.63972384_{05}^{84}	9.63972384_{04}^{44}	9.639_{55}^{73}	9.6397_{17}^{24}
2	15.19725_{184}^{201}	15.19725192_{59}^{66}	15.1972_{30}^{53}	15.197251_{75}^{93}
3	19.73920_{85}^{91}	$19.739208802178_{31}^{95}$	19.7329208_{65}^{81}	19.7392088021_{25}^{80}
4	29.52148_{04}^{18}	29.52148111_{38}^{42}	29.521_{34}^{49}	29.5214_{77}^{82}
5	31.91263_{31}^{88}	31.9126359_{37}^{59}	31.912_{18}^{64}	31.9126_{21}^{36}

High precision bound for eigenvalues over L-shaped domain

Example II: Square-minus-Square domain

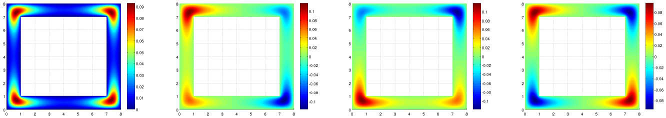
- Domain: $(0,8)^2 \setminus [1,7]^2$
- Rough estimate by applying FEM: $\lambda_5 < 35 < \lambda_6$; Degree of polynomial: 10



λ_i	lower	upper
1	9.1602158	9.1602163
2	9.1700883	9.1700889
3	9.1700883	9.1700889
4	9.1805675	9.1805681
5	10.08984333	10.08984336

Eigenvalue bounds

Eigenfunctions corresponding to leading 4 eigenvalues

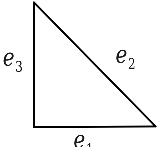


Example III: Evaluation of Babuska-Aziz constant

Babuska-Aziz constant C: For T being unit isosceles right triangle.

$$C = \frac{1}{\sqrt{\lambda_1}} : \qquad -\Delta u = \lambda u \text{ in } T, \quad \frac{\partial u}{\partial n} = c \text{ on } e_1, \quad \frac{\partial u}{\partial n} = 0 \text{ on } e_2, e_3.$$

- Value of ν: $\lambda_1 < \nu = 22.0 \le \lambda_2$.
- Degree of Bernstein polynomial: 6.



Computation result

$$4.115858165137194 \le \lambda_1 \le 4.115858370667702$$

Or, $0.492912451457115 \le C \le 0.4929124663764214$

Example IV:

Take $\Omega = (0,\pi) \times (0,\pi)$. We consider the following eigenvalue problem

$$-\operatorname{div}(p\nabla u) = \lambda u \text{ in } \Omega, u = 0 \text{ on } \partial\Omega$$

$$p := 1 + \frac{16}{\pi^4}\ x(\pi - x)y(\pi - y)$$

Base problem:

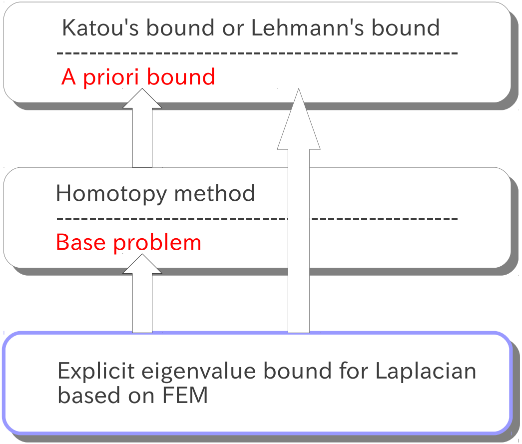
$$-\Delta u = \lambda u \text{ in } \Omega, u = 0 \text{ on } \partial\Omega$$

Computation results:

λ_i	lower	upper
1	2.703809	2.703823
2	7.2208409	7.2208468
3	7.2208437	7.2208468
4	11.1500 372	11.1500435

The bounds for the leading 4 eigenvalues [Plum, 1991]

The framework of high precision eigenvalue bounds



Reference:

- M. Plum. Bounds for eigenvalues of second-order elliptic differential operators. The Journal of Applied Mathematics and Physics(ZAMP), 42(6):848-863, 1991.