Computing the best possible pseudo-solution to interval linear system of equations

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Problem statement

Let $\mathbf{A}x = \mathbf{b}$ – linear equation set, where elements of the matrices $\mathbf{A}$ and $\mathbf{b}$ are intervals $\mathbf{a_{ij}} = \left[\underline{a}_{ij}, \overline{a}_{ij}\right]$, $\mathbf{b_j} = \left[\underline{b}_j, \overline{b}_j\right]$, $i, j = 1, 2, \ldots, n$.

AE-solution set

In accordance with classification of Kearfott, Shary, other AE-solution of interval linear equation set $\mathbf{A}x = \mathbf{b}$ are points of tolerable set

$$\Theta_{tol}(\mathbf{A}, \mathbf{b}) = \left\{ x : \ (\forall i, j = 1, 2, \ldots, n) \left(\forall a_{ij} \in \mathbf{a_{ij}}\right) \left(\sum_{j=1}^{n} a_{ij} x_j \in \mathbf{b_i} \right) \right\}, \qquad (1)$$

EE-solution set

EE-solution of considered system are points of united set

$$\Theta_{uni}(\mathbf{A}, \mathbf{b}) = \left\{ x : \ (\forall i, j = 1, 2, \ldots, n) \left(\exists a_{ij} \in \mathbf{a_{ij}}\right) \left(\sum_{j=1}^{n} a_{ij} x_j \in \ \mathbf{b_i} \right) \right\} \qquad (2)$$

Predecessors

Search of EE-solution

In Lakeyev, Kreinovich's papers was proved that search of EE-solution of interval equation set is NP-hard problem!

AE-solution

In accordance with Rohn's theorem, any point of tolerable set of AE-solution can be presented as $x = x^+ - x^-$, where x^+, x^- are solutions of set of inequalities

$$\begin{aligned}
\sum_{j=1}^{n} \left[\underline{a}_{ij} x_j^+ - \overline{a}_{ij} x_j^-\right] &\geq \underline{b}_i, \; i = 1, 2, \ldots, n., \\
\sum_{j=1}^{n} \left[\overline{a}_{ij} x_j^+ - \underline{a}_{ij} x_j^-\right] &\leq \overline{b}_i, \; i = 1, 2, \ldots, n, \\
x^+, x^- &\geq 0.
\end{aligned}$$

Consequently, search of AE-solutions is polynomial complexity problem.

Predecessors

Estimation of AE-solutions

Methods of estimation of AE-solutions for $\Theta_{tol}(\mathbf{A}, \mathbf{b}) \neq \emptyset$ cases are considered in Neumaier, Shary works and other.

≪Recognizing functional≫

≪Recognizing functional method≫founded and developed in Novosibirsk by Shary. Making decision about task solvability(i.e. about emptiness/nonemptyness of solutions set) requires manipulations with special (nonsmooth and concave) functional, named ≪recognizing≫.

Maximization of recognizing functional which may be performed with for example optimization methods, developed in Institute of cybernetics NAS of Ukraine. It gives sufficiently substantive information for possible task correction.

Software of S. P. Shary and P. I. Stetsyuk, are freeware in the web, implementation is done in INTLAB, MATLAB's interval extension, and Int4Sci, Scilab's interval extension.

Pseudo-solution

Reason

In most practical cases set of inequalities

$$\begin{aligned}\sum_{j=1}^{n}\left[\underline{a}_{ij}x_j^+ - \overline{a}_{ij}x_j^-\right] &\geq \underline{b}_i,\ i=1,2,\dots,n.,\\ \sum_{j=1}^{n}\left[\overline{a}_{ij}x_j^+ - \underline{a}_{ij}x_j^-\right] &\leq \overline{b}_i,\ i=1,2,\dots,n,\\ x^+, x^- &\geq 0.\end{aligned}$$

is ill-conditioned or even inconsistent. For this case by analogy with Tikhonov & Ivanov's works introduction of «pseudo-solution»conception is reasonable.

Definition

For given interval system we construct parametrized family of equation sets $\mathbf{A}x = \mathbf{b}(z)$ with modified right part $\mathbf{b}(z) = \left[\underline{b} - z\left|\underline{b}\right|, \overline{b} + z\left|\overline{b}\right|\right], z \geq 0$.
Let $z^* = \inf\{z : \Theta_{tol}(\mathbf{A}, \mathbf{b}(z)) \neq \emptyset\}$. **Pseudo-solution** of basic system $\mathbf{A}x = \mathbf{b}$ we call inner points of tolerable set $\Theta_{tol}(\mathbf{A}, \mathbf{b}(z^*))$.

Correctness of the definition

Validation of introduced definition gives theorem below.

Theorem

For any interval equation set $\mathbf{A}x = \mathbf{b}$ for all $z \geq 1$ set $\Theta_{tol}(\mathbf{A}, \mathbf{b}(z)) \neq \emptyset$.

Доказательство.

For set of inequalities (slide above) put in inequalities $x^+ = x^- = 0$, this yields

$$0 \geq \underline{b}_i - z|\underline{b}_i|, \quad 0 \leq \overline{b}_i + z|\overline{b}_i|, \quad i = 1, 2, \ldots, n. \tag{3}$$

Thus, for all$z \geq 1$ inclusion $0 \in \Theta_{tol}(\mathbf{A}, \mathbf{b}(z))$ is true. Theorem is proved. □

Pseudo-solution retrieving

Way to find a pseudo-solution of system of equation $\mathbf{A}x = \mathbf{b}$ is given by

Theorem

Exist solution ${x^+}^*,\ {x^-}^* \in \mathrm{R}^n$, $z^* \in \mathrm{R}$ of linear programming task

$$z \to \min_{x^+,\ x^-,\ z}, \tag{4}$$

$$\sum_{j=1}^{n} (\underline{a}_{ij} x_j^+ - \overline{a}_{ij} x_j^-) \geq \underline{b}_i - z|\underline{b}_i|, \quad i = 1, 2, \ldots, n, \tag{5}$$

$$\sum_{j=1}^{n} (\overline{a}_{ij} x_j^+ - \underline{a}_{ij} x_j^-) \leq \overline{b}_i + z|\overline{b}_i|, \ i = 1, 2, \ldots, n, \tag{6}$$

$$x_j^+, x_j^-, z \geq 0, \quad j = 1, 2, \ldots, n, \tag{7}$$

besides $x^* = {x^+}^* - {x^-}^*$ is pseudo-solution of system $\tilde{A}x = \tilde{b}$

Obstructions of realization

Introduced conception of ≪pseudo-solution ≫of interval equation set is fully constructive and make it possible to give the result for the interval system with empty tolerable set $\Theta_{tol}(\mathbf{A}, \mathbf{b})$.

Obstructions of simplex method

Linear programming task is strong degenerate, and its solving with usage of standard floating point data types is impossible in common case because of cycling is not eliminated by known anticycling tools under approximate computation.

Way to realization

Looping may be disabled at the expense of using exact calculations, it provides, for example, ≪Exact Computational ≫library or GNU MPA library.

Complexity of exact solution

In this case each iteration of the simplex method requires no more than $4lm^4 + O(lm^3)$ bit of memory, where m – minimum from task dimensions, l – number of bit sufficient for storage one element of input matrix, also parallel efficiency (i.e. fraction speedup to processors number) is asymptotic value near 100 %.

Some realization moments

Components

Effective solution consist of two main components

- Effective exact computations
- Balanced palallel linear programming task decomposition

We use modern version of the ≪Exact Computatonal≫library, optimized for x86 and x64 processors, it storages numbers with 2^{32} radix, that makes the numbers shorter and the arithmetic operations faster. There are a lot of groundwork of parallel basic arithmetic operations, including heterogenious(CPU+GPU) computational environments. All developed types have standard interface and my be used as embedded C++ data types.

The simplex method parallel realization technics fully described in Panyukov and Gorbik's work in ≪Automatics and telemechanics ≫journal No. 2, 2012 year. The realization of this technics is in the pipeline now.

Computing experiment

In the computing experiment Intel Core i7-950 3.06 GGh, 6 Gb RAM, GPU Nvidia 460(1Gb GDDR5) system, under Win 7 x64 OS, 64-bit Visual C++ 2011 compiler was used.

As model task we used interval system:

$$\mathbf{A} = \left[\frac{i*(1-\delta)}{i+j-1}, \frac{i*(1+\delta)}{i+j-1}\right]_{n\times n}; \ \mathbf{b} = [1, 1/2, \dots, 1/(n-1), 1/n]^T$$

The dependency of minimal extension of right part (z^* parameter), corresponding to pseudo-solution, with fixed $n = 20$ is shown in table

δ	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
z^*	0.81	0.389	0.1	0.025	0.0062	0.0017

Table below shows time result for different task size.

Matrix size (n)	10	20	50	100
Work time	0.46s	7.73s	7.39m	15.1h

Summary

- Pseudo-solution of interval linear equation set was introduced
- Some technics of program realization was introduced.
- Computing experiment's results for different modeling input data are granted.

Computing experiment shows that more effective and parallel realization are needed. Further work will be concentrated on more effective realizations by optimization of the parallel version of program.

Thank you for attention!