Algorithm for Sparse Approximate Inverse Preconditioners Refinement in Conjugate Gradient Method

Labutin Ilya
Institute of Petroleum Geology and Geophysics SB RAS
Surodina Irina
Institute of Computational Mathematics and Mathematical Geophysics

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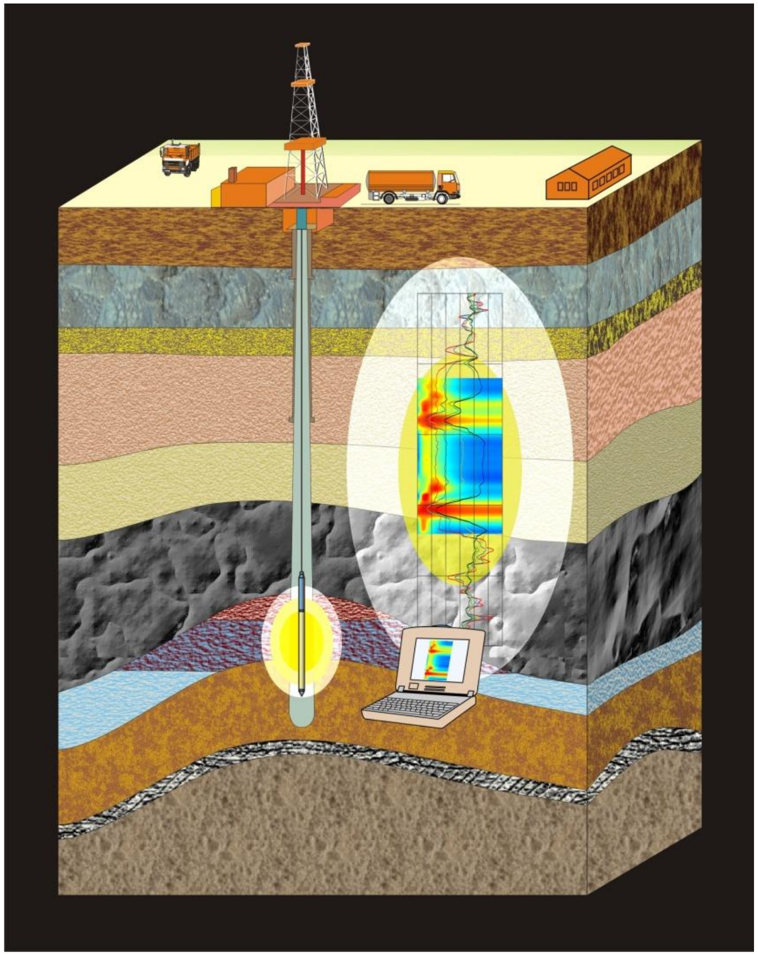
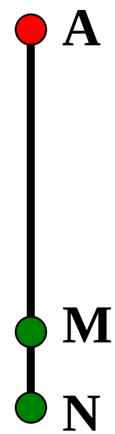
Agenda

- Motivation
- Parallel preconditioners
- Numerical experiments

Well resistivity logging





GZ1 – A0,4M0,1N
GZ2 – A1M0,1N
GZ3 – A2M0,5N
GZK – N0,5M2A
GZ4 – A4M0,5N

Potential field problem

$$\frac{1}{r}\frac{\partial}{\partial r}(\sigma \; r\,\frac{\partial U^{a}}{\partial r})+\frac{\partial}{\partial z}(\sigma \; \frac{\partial U^{a}}{\partial z})= \frac{1}{r}\frac{\partial}{\partial r}((\sigma_0-\sigma \;)\,r\,\frac{\partial U^{0}}{\partial r})+\frac{\partial}{\partial z}((\sigma_0-\sigma \;)\frac{\partial U^{0}}{\partial z})$$

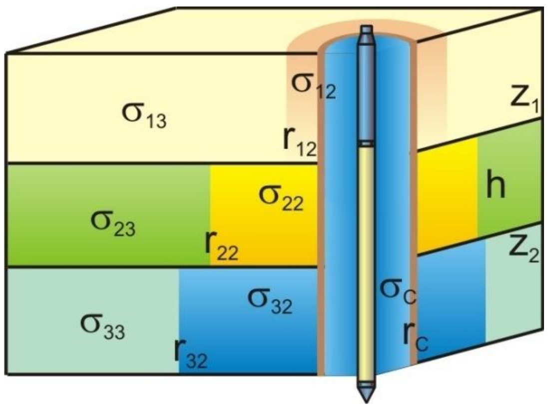
$$\hbar_i^{(r)}=\left(h_i^{(r)}+h_{i+1}^{(r)}\right)/2 \qquad h_i^{(r)}=r_i-r_{i-1} \qquad i=1,\ldots,N_r \qquad \left(V\right)_{\bar{r},ij}=\left(V_{ij}-V_{i-1,j}\right)/\,h_i^{(r)}$$

$$a_{ij}=\sigma(r_i-\frac{h_i^{(r)}}{2},z_j+\frac{h_j^{z}}{2}) \qquad b_{ij}=\sigma(r_i+\frac{h_i^{(r)}}{2},z_j-\frac{h_j^{(z)}}{2})$$

$$Ax=F$$

$$AV=-\frac{1}{r}\left(\bar{r}\,aV_{\bar{r}}\right)_{\hat{r}}-\left(bV_{\bar{z}}\right)_{\hat{z}}$$

$$F=\frac{1}{r}\left(\bar{r}\,(a-\sigma_0)U_{\bar{r}}^{0}\right)_{\hat{r}}+\left((b-\sigma_0)U_{\bar{z}}^{0}\right)_{\hat{z}}-\frac{(a-\sigma_0)}{r^2}U^{0}$$



r_{jl} - Radial bounds

z_j - Vertical bounds

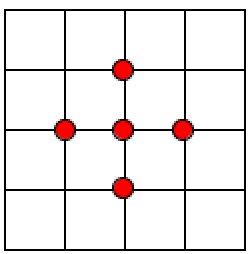
σ_{jl} - Conductivity

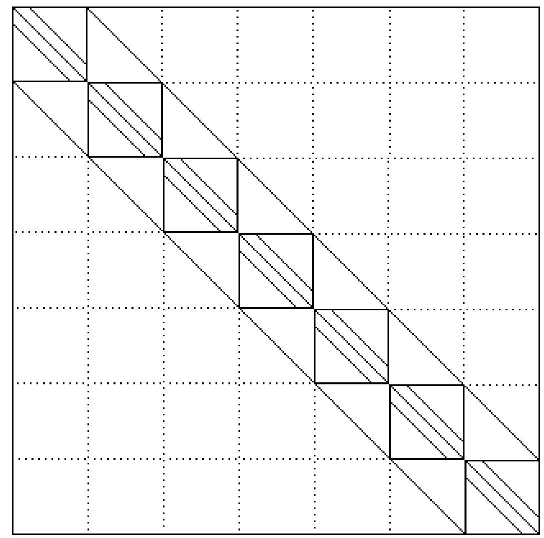
Дашевский Ю.А., Суродина И.В. Эпов М.И. Квази-трехмерное математическое моделирование диаграмм неосесимметричных зондов

Finite difference approximation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\sigma\ r\frac{\partial U^{a}}{\partial r}\right)+\frac{\partial}{\partial z}\left(\sigma\ \frac{\partial U^{a}}{\partial z}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left((\sigma_{0}-\sigma\ )\, r\frac{\partial U^{0}}{\partial r}\right)+\frac{\partial}{\partial z}\left((\sigma_{0}-\sigma\ )\frac{\partial U^{0}}{\partial z}\right)$$

$$AV=-\frac{1}{r}\left(\bar{r}\,aV_{\bar{r}}\right)_{\hat{r}}-\left(bV_{\bar{z}}\right)_{\hat{z}}$$





$$k(A)=\lambda_{max}/\lambda_{min}$$

- Sparse
- Symmetric
- Positive definite
- Not strictly diagonally dominant matrix

Conjugate gradient method

$k = 0: Initialization:\ x_0, p_0 = r_0 = b - Ax_0$

$k \geq 0: While\, \frac{\|r_k\|}{\|r_0\|} > \varepsilon$

1. $q_k = Ap_k, \alpha_k = \frac{\|r_k\|^2}{{p_k}^T q_k}$
2. $x_{k+1} = x_k + \alpha_k p_k, r_{k+1} = r_k - \alpha_k q_k$
3. $\beta_k = \frac{\|r_{k+1}\|^2}{\|r_k\|^2}, p_{k+1} = r_{k+1} + \beta_k p_k$

- Scalar product
- Norm
- Vector updates
- Matrix vector product

Preconditioned conjugate gradient method

$M^{-1}Ax = M^{-1}b$

$A\,M^{-1}y{=}b,\ x{=}M^{-1}y$

$k = 0: Initialization:\ x_0, r_0 = b - Ax_0, Mz_0 = r_0, p_0 = z_0$

$k \geq 0: While\,\frac{\|r_k\|}{\|r_0\|} > \varepsilon$

1. $q_k = Ap_k, \alpha_k = \frac{{z_k}^T r_k}{{p_k}^T q_k}$
2. $x_{k+1} = x_k + \alpha_k p_k, r_{k+1} = r_k - \alpha_k q_k$
3. $Mz_{k+1} = r_{k+1}$
4. $\beta_k = \frac{{z_{k+1}}^T r_{k+1}}{{z_k}^T z_k}, p_{k+1} = r_{k+1} + \beta_k p_k$

- Need to solve additional linear system at each step
- Additional costs should not outweigh reduction of iterations
- Classical preconditioners (SOR, Incomplete Factorizations) are based on triangular decompositions – sequential task
- Simple preconditioners like Jacobi has limited impact on the efficiency
- Sparse Approximate Inverse – sequential setup phase
- Incomplete Poisson Preconditioner (***M.Ament, G.Knittel, A Parallel Preconditioned Conjugate Gradient Solver for the Poisson Problem on a Multi-GPU Platform***)

Schulz-Hotelling algorithm

$D_0 -$ initial approximation of A^{-1}

With $\|R_0\| \leq q \leq 1$, where $R_0 = I - AD_0$, we can build iterative process

$D_1 = D_0(I + R_0), R_1 = I - AD_1$

$D_2 = D_1(I + R_1), R_2 = I - AD_2$

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$D_m = D_{m-1}(I + R_{m-1}), R_m = I - AD_m$

$$\|D_m - A^{-1}\| \leq \|D_0\| \frac{q^{2^m}}{1-q}$$

If $A = A^T$ and $D_0 = D_0{}^T$, then $D_m = D_m{}^T$

G. Schulz, Iterative Berechnung der reziproken Matrix, Z. Angew, Math. Mech. 13 (1933)
H. Hotelling, Analysis of a complex of statistical variables into principal components, J.Educ.Psych.,(1933)

Schulz-Hotelling algorithm

$$D_1 = D_0(I + R_0)$$

$$D_2 = D_0(I + R_0 + R_0^2 + R_0^3)$$

$$D_3 = D_0(I + R_0 + R_0^2 + R_0^4 + R_0^5 + R_0^6 + R_0^7)$$

$$D_m = D_0(I + R_0 + R_0^2 + \ldots + R_0^{2^m - 1})$$

Parallel preconditioners (Jacobi)

Initial inverse approximation:
$D_0 = diag\{{a_{11}}^{-1}, {a_{22}}^{-1}, \dots, {a_{nn}}^{-1}\}$

$D_1 = D_0 + D_0(I - AD_0)$ – symmetric, 5-diagonal

$D_2 = D_1 + D_1(I - AD_1)$ – symmetric, 25-diagonal

$D_2 = 2D_1 + D_1AD_1$

$D_2 = D_1\left(I + {R_0}^2\right), \qquad R_0 = I - AD_0$

$I + {R_0}^2$ - 9-diagonal

$D_3 = D_2 + D_2(I - AD_2) = 2(2D_1 - D_1AD_1) - (2D_1 - D_1AD_1)A(2D_1 - D_1AD_1)$

$D_3 = 2D_1\left(I + {R_0}^2\right) - D_1\left(I + {R_0}^2\right)AD_1\left(I + {R_0}^2\right)$-symmetric, 113-diagonal

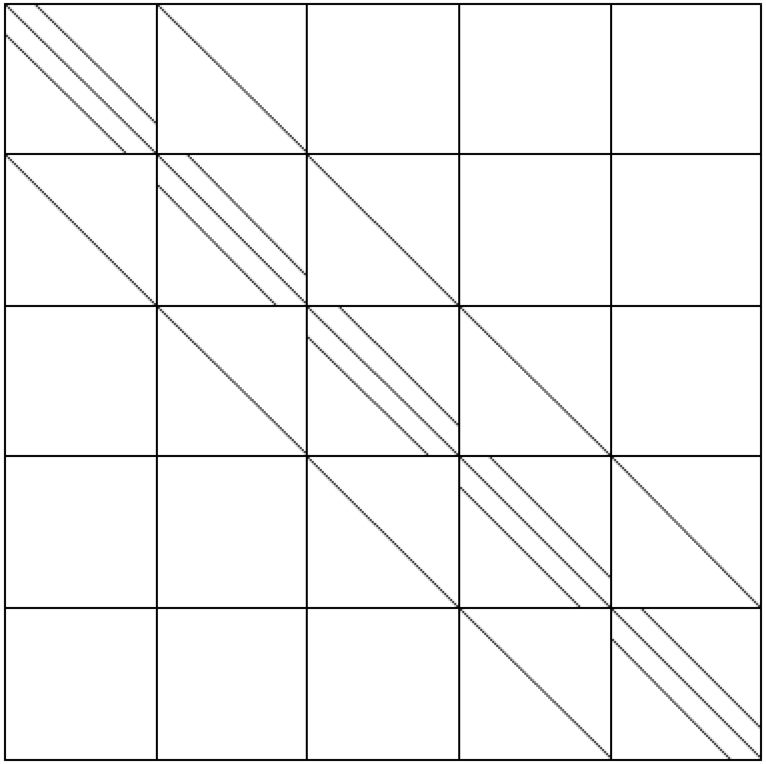
Parallel preconditioners (D_1)

$$D_1 = D_0 + D_0(I - AD_0)$$

$$D_1 3(i) = 1/A3(i)$$

$$D_1 4(i) = -A4(i)/(A3(i+1) * A3(i))$$

$$D_1 5(i) = -A5(i)/(A3(i+1) * A3(i))$$

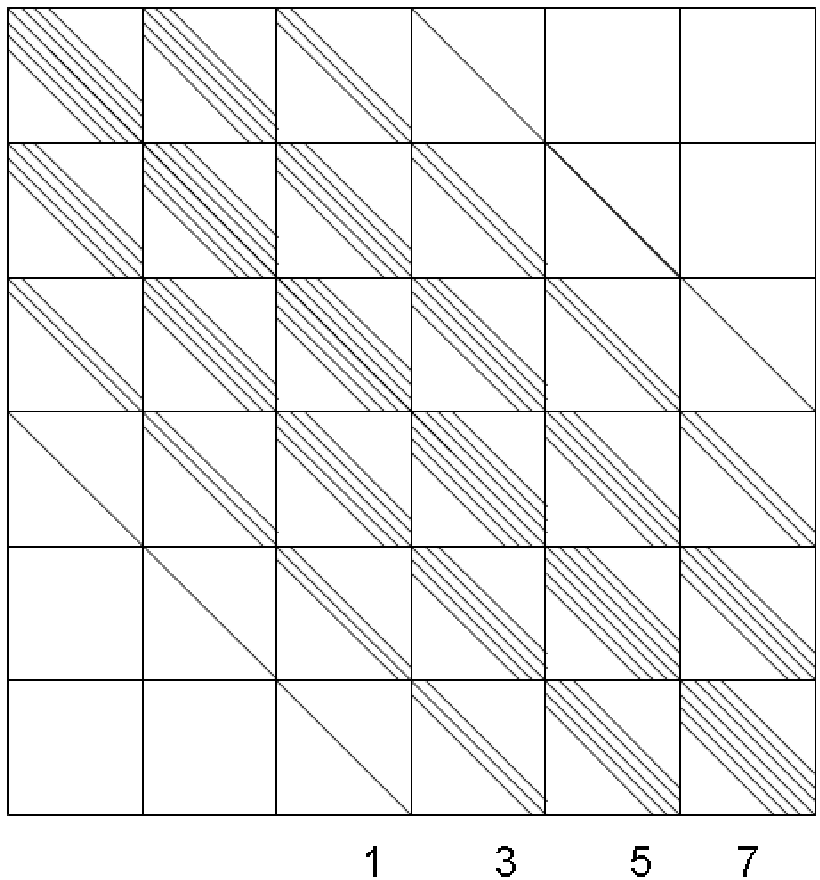


Parallel preconditioners (D_2)

$D_2 = D_1 + D_1(I - AD_1)$ – symmetric, 25-diagonal

$\boldsymbol{D_2 = 2D_1 + D_1 A D_1}$

$D_2 = D_1\left(I + {R_0}^2\right), \quad R_0 = I - AD_0, I + {R_0}^2$ - 9-diagonal



Parallel preconditioners (D_3)

$D_3 = D_2 + D_2(I - AD_2) = 2(2D_1 - D_1AD_1) - (2D_1 - D_1AD_1)A(2D_1 - D_1AD_1)$

$D_3 = 2D_1(I + {R_0}^2) - D_1(I + {R_0}^2)AD_1(I + {R_0}^2)$-symmetric, 113-diagonal

15	13	11	9	7	5	3	1		
13	15	13	11	9	7	5	3	1	
11	13	15	13	11	9	7	5	3	1
9	11	13	15	13	11	9	7	5	3
7	9	11	13	15	13	11	9	7	5
5	7	9	11	13	15	13	11	9	7
3	5	7	9	11	13	15	13	11	9
1	3	5	7	9	11	13	15	13	11
	1	3	5	7	9	11	13	15	13
		1	3	5	7	9	11	13	15

Parallel preconditioners (SSOR)

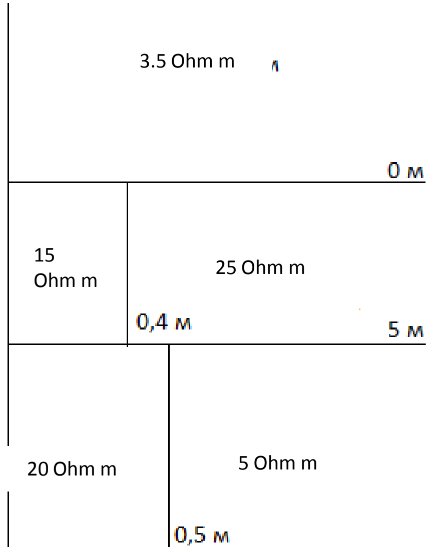
$A = L + D + L^T, D -$ diagonal, $L -$ lower triangular
$M = KK^t$, where $K = \frac{(\overline{D}+L)\overline{D}^{-1/2}}{\sqrt{2-\omega}}, 0 < \omega < 2, \overline{D} = \left(\frac{1}{\omega}\right) D$
$K^{-1} = \sqrt{2 - \omega\overline{D}}(I + \overline{D}^{-1}L)^{-1}\overline{D}^{-1}$
$K^{-1} \approx \sqrt{2 - \omega}\overline{D}^{\frac{1}{2}}[I - \overline{D}^{-1}L + (\overline{D}^{-1}L)^2 - (\overline{D}^{-1}L)^3 + \cdots]\overline{D}^{-1}$
$\overline{K} = \sqrt{2 - \omega}\overline{D}^{\frac{1}{2}}(I - \overline{D}^{-1}L)\overline{D}^{-1} = \sqrt{2 - \omega}\overline{D}^{-\frac{1}{2}}(I - L\overline{D}^{-1})$
SSOR-AI preconditioner is defined as $\overline{M} = \overline{K}^T\overline{K}$

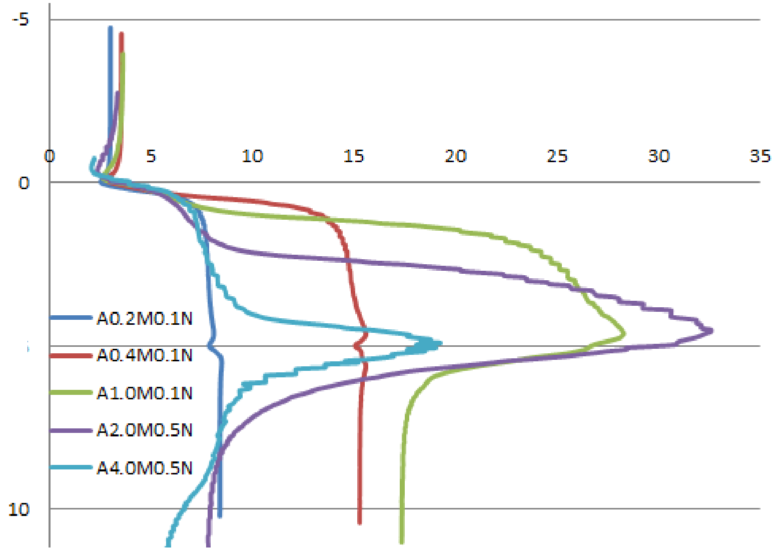
Let's take as D_0 in Schulz-Hotelling series

*-***Rudi Helfenstein, Jonas Koko, Parallel preconditioned conjugate gradient algorithm on GPU***

Numeric experiments

Probes: 5
Tool positions: 100





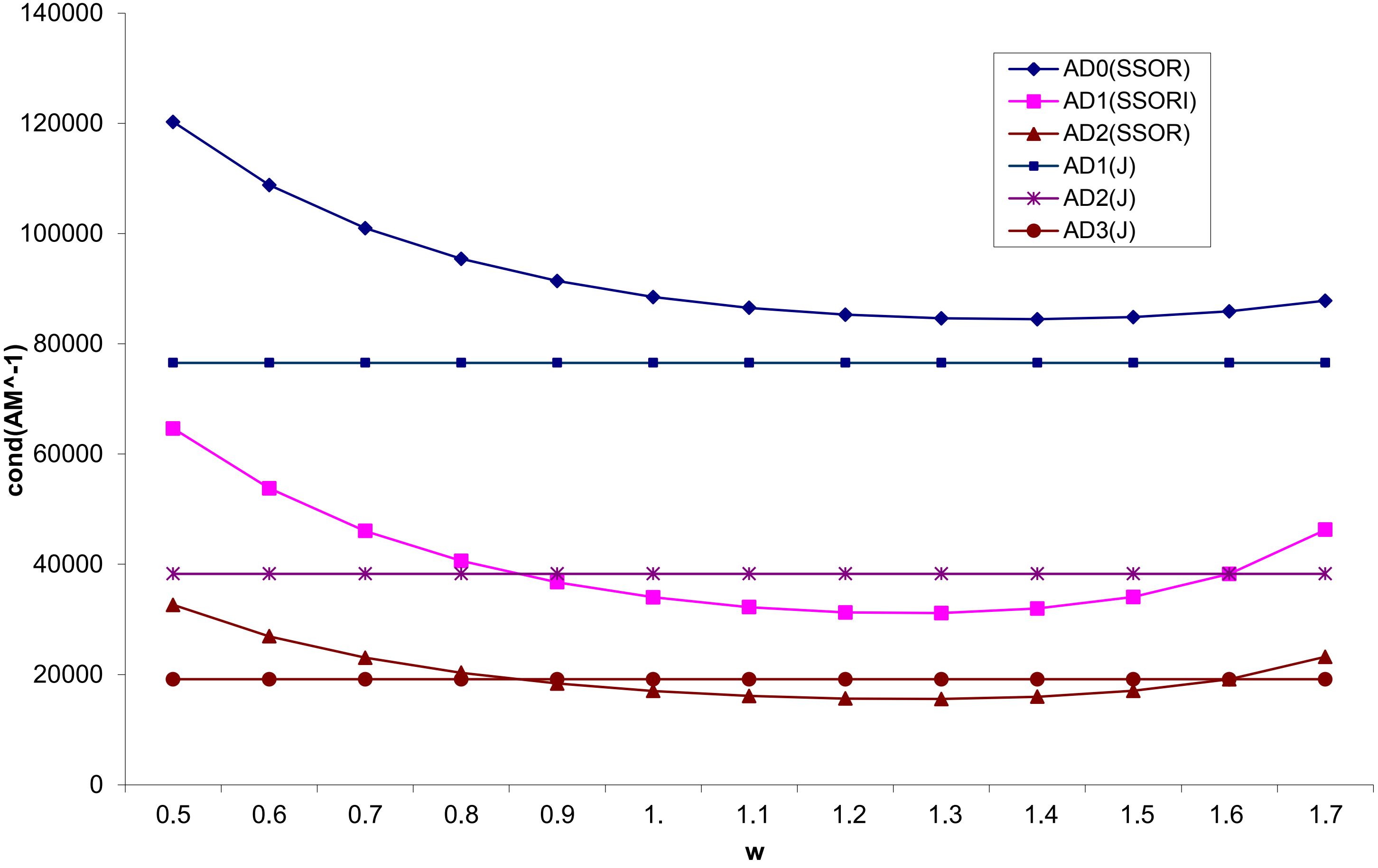
Grid: 88x200, matrix 17600x17600
GPU: NVIDIA® GeForce® GTX 480

Condition number of AM^{-1}

Matrix	Condition number
A	$4.5377 * 10^7$
AJ	$3.0542 * 10^5$
$AD_1(J)$	$7.6542 * 10^4$
$AD_2(J)$	$3.8271 * 10^4$
$AD_3(J)$	$1.9135 * 10^4$
$AD_0(SSOR)$	$8.4605 * 10^4$
$AD_1(SSOR)$	$3.1132 * 10^4$
$AD_2(SSOR)$	$1.5556 * 10^4$

Grid: 88x200, matrix 17600x17600
GPU: NVIDIA® GeForce® GTX 480

Condition number of AM^{-1}



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Performance

Method	Iterations	Time, sec	Method	Iterations	Time, sec
CG	219322	62,7			
$D_0 = J$	2428	0.477			
$D_1(J)$	1209	0.188	$D_0 = SSOR$	1309	0.228
$D_2(D_1)$	859	0.178	$D_1(SSOR)$	784	0.177
$D_2(R^2)$	859	0.174			
$D_3(D_1)$	608	0.172	$D_2(SSOR)$	554	0.179

	$D_3(D_1)$	$D_1(SSOR)$	$D_2(SSOR)$
Time, sec	14	17	15

Grid: 88x200, matrix 17600x17600

GPU: NVIDIA® GeForce® GTX 480

Conclusion

- A parallel preconditioner is presented
- A sparse approximate inverse is computed explicitly
- Computation of the preconditioner is inherently parallel (well suitable for GPU)

Thank you