Computing Interval Power Functions

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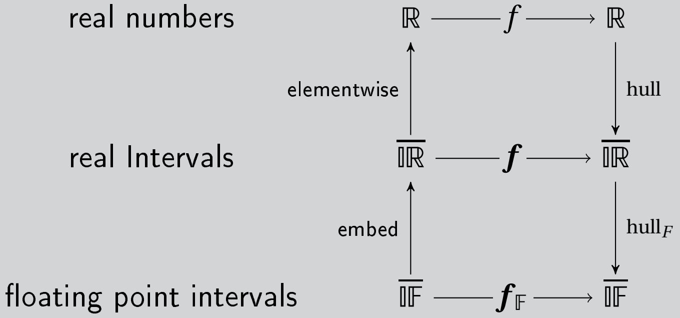
Interval Functions

Evaluation



Interval Extension

Definition

Every interval function $\boldsymbol{f}$ that contains the range $f(\boldsymbol{x})$ is called an interval extension of f

$$x \in \boldsymbol{x} \Rightarrow f(x) \in \boldsymbol{f}(\boldsymbol{x}), \text{ if } f(x) \text{ is defined}$$

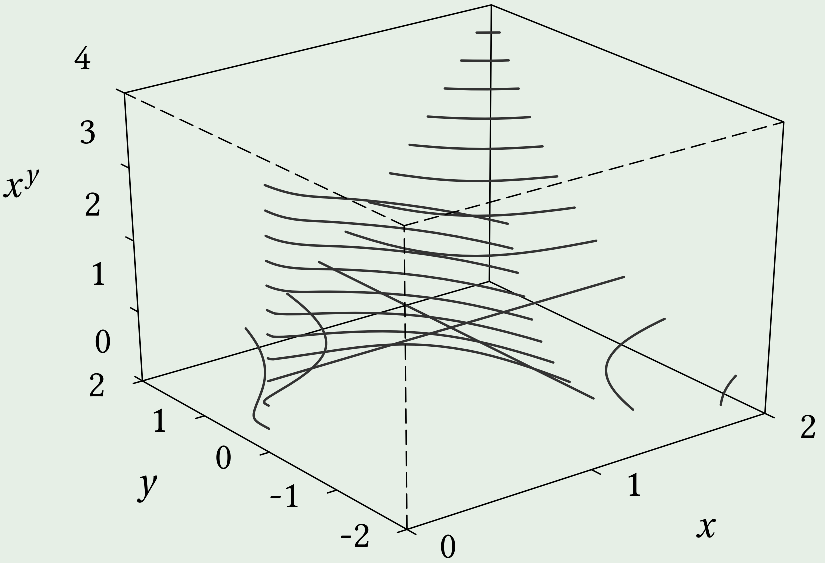
FTIA

Every interval version of an arithmetic expression is an interval extension of the function defined by that expression.

Example

Naive interval evaluation of an arithmetic expression.

General exponential function x^y



Enlarging the Domain

$\mathrm{pow}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

1. positive integral exponents $x^n = x \cdot \ldots \cdot x$
2. integral exponents $x^{-n} = 1/x^n$
3. rational exponents $x^{m/n} = \sqrt[n]{x}^m$
4. real exponents $x^y = \exp(y \cdot \log x)$

Problems

- 0^0?
- $\sqrt[n]{x}^m \in \mathbb{R}$ if $x < 0$?
- $\exp(y \cdot \log x) \in \mathbb{R}$ if $x \leq 0$?

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Basic function pow1

$$\text{pow1} \quad \mathbb{R}^{+} \times \mathbb{R} \quad (x, y) \mapsto \exp(y \cdot \log(x))$$

pow1: positive Version

- Definition only for $x > 0$
- sufficient for many applications
- IEEE 754 *pow* and *powr*
- mathematical well founded
- differentiable
- restricted, smooth domain and range

minimal Extension pow1

pow1	$\mathbb{R}^+ \times \mathbb{R}$	$(x, y) \mapsto \exp(y \cdot \log x)$
powzero	$\{0\} \times \mathbb{R}^+$	$(x, y) \mapsto 0$

minimal Extension pow1

$$\mathbb{R}^+ \times \mathbb{R} \quad (x, y) \mapsto \exp(y \cdot \log x)$$

powzero $$\{0\} \times \mathbb{R}^+ \quad (x, y) \mapsto 0$$

pow2 Extension 1 of pow1

pow1	$\mathbb{R}^+ \times \mathbb{R}$	$(x, y) \mapsto \exp(y \cdot \log x)$
pow2	$\{0\} \times \mathbb{R}^+$	$(x, y) \mapsto 0$
	$\mathbb{R}^- \times \mathbb{Z}$	$(x, y) \mapsto \begin{cases} \exp(y \cdot \log \lvert x \rvert) & \text{if } y \text{ even} \\ -\exp(y \cdot \log \lvert x \rvert) & \text{if } y \text{ odd} \end{cases}$

pow2 Extension 1 of pow1

$$\text{pow2} \quad \begin{array}{ll} \mathbb{R}^{+} \times \mathbb{R} & (x,y) \mapsto \exp(y \cdot \log x) \\ \{0\} \times \mathbb{R}^{+} & (x,y) \mapsto 0 \\ \mathbb{R}^{-} \times \mathbb{Z} & (x,y) \mapsto \begin{cases} \exp(y \cdot \log|x|) & \text{if } y \text{ even} \\ -\exp(y \cdot \log|x|) & \text{if } y \text{ odd} \end{cases} \end{array}$$

Variants of pow

pow2: limited Version

- comprises all (commonly agreed) cases of real exponentiation
- restricted, partially discrete domain and range

Variants of pow

complex Version

- uses principal branch of complex logarithm and complex exponential function
- $\exp(y \cdot \log x) \in \mathbb{C}$
- well understood in pure mathematics
- not defined for $x = 0$, continuous for $y > 0$
- as complex function not suitable
- But: restriction to real domain yields pow2

pow3 Extension 2 of pow1

pow1	$\mathbb{R}^+ \times \mathbb{R}$	$(x, y) \mapsto \exp(y \cdot \log x)$
	$\{0\} \times \mathbb{R}^+$	$(x, y) \mapsto 0$
pow3	$\mathbb{R}^- \times \mathbb{Q}_{odd}$	$(x, \frac{m}{n}) \mapsto \begin{cases} \lvert x\rvert^{m/n} & \text{if } m \text{ even} \\ -\lvert x\rvert^{m/n} & \text{if } m \text{ odd} \end{cases}$

$\mathbb{Q}_{odd}$: odd denominators

pow3 Extension 2 of pow1

$$\mathbb{R}^+ \times \mathbb{R} \quad (x, y) \mapsto \exp(y \cdot \log x)$$

$$\{0\} \times \mathbb{R}^+ \qquad (x, y) \mapsto 0$$

pow3 $$\mathbb{R}^- \times \mathbb{Q}_{odd} \quad (x, \tfrac{m}{n}) \mapsto \begin{cases} |x|^{m/n} & \text{if } m \text{ even} \\ -|x|^{m/n} & \text{if } m \text{ odd} \end{cases}$$

$\mathbb{Q}_{odd}$: odd denominators

Variants of pow

pow3: extended Version

- for $x < 0$ NOT continuous
- very crisp domain
- contentious application, but may be helpful

Definition of Interval Functions

$$\mathbf{pow}1(\boldsymbol{x},\boldsymbol{y}) = \mathrm{hull}\left(\{\mathrm{pow}1(x,y) \mid x \in \boldsymbol{x} \text{ und } y \in \boldsymbol{y}\}\right)$$
$$\mathbf{pow}2(\boldsymbol{x},\boldsymbol{y}) = \mathrm{hull}\left(\{\mathrm{pow}2(x,y) \mid x \in \boldsymbol{x} \text{ und } y \in \boldsymbol{y}\}\right)$$
$$\mathbf{pow}3(\boldsymbol{x},\boldsymbol{y}) = \mathrm{hull}\left(\{\mathrm{pow}3(x,y) \mid x \in \boldsymbol{x} \text{ und } y \in \boldsymbol{y}\}\right)$$

pow1 straight-forward

$$\mathrm{pow1}(x,y) = \exp(y \cdot \log x)$$

$$\begin{aligned}
[\underline{l}, \overline{l}] &= \mathbf{log}_{\mathbb{F}}[\underline{x}, \overline{x}] \\
&= [\nabla \log \underline{x}, \triangle \log \overline{x}], \\
[\underline{m}, \overline{m}] &= [\underline{y}, \overline{y}] \bullet_{\mathbb{F}} [\underline{l}, \overline{l}] \\
&= [\min\{\nabla(\underline{y} \cdot \underline{l}), \nabla(\underline{y} \cdot \overline{l}), \nabla(\overline{y} \cdot \underline{l}), \nabla(\overline{y} \cdot \overline{l})\}, \\
&\quad\ \max\{\triangle(\underline{y} \cdot \underline{l}), \triangle(\underline{y} \cdot \overline{l}), \triangle(\overline{y} \cdot \underline{l}), \triangle(\overline{y} \cdot \overline{l})\}], \\
[\underline{z}, \overline{z}] &= \mathbf{exp}_{\mathbb{F}}[\underline{m}, \overline{m}] \\
&= [\nabla \exp \underline{m}, \triangle \exp \overline{m}],
\end{aligned}$$

Distinction of Cases

Goal

Reduce number of operations in floating-point

Approach

Distinction of cases

- $1 \in \boldsymbol{x}$?
- $0 \in \boldsymbol{y}$?

pow1 improved evaluation

The value of $\mathbf{pow1}([\underline{x},\overline{x}],[\underline{y},\overline{y}])$ with $0 < \underline{x}$

	$\overline{x} \le 1$	$\underline{x} < 1 < \overline{x}$	$1 \le \underline{x}$
$0 \le \underline{y}$	$[\underline{x}^{\overline{y}}, \overline{x}^{\underline{y}}]$	$[\underline{x}^{\overline{y}}, \overline{x}^{\overline{y}}]$	$[\underline{x}^{\underline{y}}, \overline{x}^{\overline{y}}]$
$\underline{y} < 0 < \overline{y}$	$[\underline{x}^{\overline{y}}, \underline{x}^{\underline{y}}]$	$\mathrm{hull}([\underline{x}^{\overline{y}}, \underline{x}^{\underline{y}}] \cup [\overline{x}^{\underline{y}}, \overline{x}^{\overline{y}}])$	$[\overline{x}^{\underline{y}}, \overline{x}^{\overline{y}}]$
$\overline{y} \le 0$	$[\overline{x}^{\overline{y}}, \underline{x}^{\underline{y}}]$	$[\overline{x}^{\underline{y}}, \underline{x}^{\underline{y}}]$	$[\overline{x}^{\underline{y}}, \underline{x}^{\overline{y}}]$

Result

Speed-up vs. Intlab

- about 45 %
- (Intlab v6; Athlon 64 X2 4850e)

pow1 improved evaluation

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$\underline{y} < 0 < \overline{y}$	$[\underline{x}^{\overline{y}}, \underline{x}^{\underline{y}}]$	$\mathrm{hull}([\underline{x}^{\overline{y}}, \underline{x}^{\underline{y}}] \cup [\overline{x}^{\underline{y}}, \overline{x}^{\overline{y}}])$	$[\overline{x}^{\underline{y}}, \overline{x}^{\overline{y}}]$
$\overline{y} \leq 0$	$[\overline{x}^{\overline{y}}, \underline{x}^{\underline{y}}]$	$[\overline{x}^{\underline{y}}, \underline{x}^{\underline{y}}]$	$[\overline{x}^{\underline{y}}, \underline{x}^{\overline{y}}]$

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Accuracy not tightest

Lemma

Let $[\underline{x}, \overline{x}], [\underline{y}, \overline{y}] \in \overline{\mathbb{ID}}$ *where* $\mathbb{D}$ *is IEEE-754 binary64 format.*
Let

$$[\underline{z}, \overline{z}] = \mathbf{exp}_{\mathbb{D}}([\underline{y}, \overline{y}] \bullet_{\mathbb{D}} \mathbf{log}_{\mathbb{D}}[\underline{x}, \overline{x}]).$$

then each of the normal, finite interval boundaries $\underline{z}$ *or* $\overline{z}$ *has a worst-case relative error of* $\tilde{\varepsilon} = 2^{-41}$ *compared to the exact boundary.*

improvement needed

- Lauter, Lefevre
- crlibm

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Evaluation of **pow**2

$$\mathrm{pow2} : (\mathbb{R}^+ \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}^+) \cup (\mathbb{R}^- \times \mathbb{Z}) \to \mathbb{R},$$

$$(x, y) \mapsto \begin{cases} \exp(y \cdot \log x) & \text{if } x \text{ positive,} \\ 0 & \text{if } x \text{ zero,} \\ \exp(y \cdot \log|x|) & \text{if } x \text{ negative and } y \text{ even,} \\ -\exp(y \cdot \log|x|) & \text{if } x \text{ negative and } y \text{ odd.} \end{cases}$$

$$\mathbf{pow}2 : (\overline{\mathbb{IR}} \times \overline{\mathbb{IR}}) \to \overline{\mathbb{IR}},$$

$$(\boldsymbol{x}, \boldsymbol{y}) \mapsto \text{hull } \mathrm{pow2}(\boldsymbol{x}, \boldsymbol{y})$$

pow2 based on pow1

pow1 → pow2

- watch for integral exponents
- negative for odd exponent

pow3 similar

pow1 → pow3

- $\text{pow3}(x, \frac{m}{n}) = \begin{cases} |x|^{m/n} & \text{if } m \text{ even} \\ -|x|^{m/n} & \text{if } m \text{ odd} \end{cases}$
  $(x < 0,\ n \text{ odd})$
- for negative bases positive and negative powers on dense subsets
- interval extension “wipes out” sign

examples

x^y	**pow1**	**pow2**	**pow3**
$[3,3]^{[2,2]}$	$[9,9]$	$[9,9]$	$[9,9]$
$[-3,-3]^{[2,2]}$	$\emptyset$	$[9,9]$	$[9,9]$
$[-3,2]^{[2,2]}$	$[0,4]$	$[0,9]$	$[0,9]$
$[-3,-3]^{[2,4]}$	$\emptyset$	$[-27,81]$	$[-81,81]$
$[-3,2]^{[-2,3]}$	$[0,+\infty]$	$[-\infty,+\infty]$	$[-\infty,+\infty]$
$[-3,0]^{[-2,3]}$	$[-\infty,+\infty]$	$[-\infty,+\infty]$	$[-\infty,+\infty]$
$[0,2]^{[-2,3]}$	$[0,\infty]$	$[0,+\infty]$	$[0,+\infty]$
$[-3,2]^{[-2,0]}$	$[1/4,+\infty]$	$[-\infty,+\infty]$	$[-\infty,+\infty]$
$[-3,2]^{[0,3]}$	$[0,8]$	$[-3,8]$	$[-27,+27]$
$[-9,-9]^{[1/2,1/2]}$	$\emptyset$	$\emptyset$	$\emptyset$
$[-8,-8]^{[\nabla(1/3),\Delta(1/3)]}$	$\emptyset$	$\emptyset$	$\supseteq [-2,2]$

Conclusion

- Discussion of 3 or 4 exponential function(s)
- mathematicly founded
- algorithms for all variants but
- extended version as option
- Proof of Concept: Reference implementation
  http://exp.ln0.de/
- Marco's talk : reverse mode

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pow3 as multi-valued function

Simplifying domain pow3

- dual-valued function $\widetilde{\text{pow3}} : (x, y) \mapsto \{\pm|x|^y\}$ für $x < 0$ und alle $y \in \mathbb{R}$.
- easy to calculate
- Interval extension using **pow1**
- Difference between **pow**3 und $\widetilde{\textbf{pow}3}$ only for point-intervals $\boldsymbol{y} = [y, y] = \{y\}$

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$$\widetilde{\text{pow3}} : (\mathbb{R}^+ \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}^+) \cup (\mathbb{R}^- \times \mathbb{R}) \to \wp(\mathbb{R})$$

$$(x, y) \mapsto \begin{cases} \{\exp(y \cdot \log x)\} & \text{if } x \text{ positive,} \\ \{0\} & \text{if } x \text{ zero,} \\ \{\pm \exp(y \cdot \log|x|)\} & \text{if } x \text{ negative,} \end{cases}$$

Lemma

For negative base intervals $\boldsymbol{x}$ and exponent intervals $\boldsymbol{y}$ it holds $\widetilde{\mathbf{pow}3}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{hull}\{\pm\mathbf{pow}1(-\boldsymbol{x},\boldsymbol{y})\} = \mathrm{hull}(-\mathbf{pow}1(-\boldsymbol{x},\boldsymbol{y}) \cup \mathbf{pow}1(-\boldsymbol{x},\boldsymbol{y}))$.

Lemma

For base intervals $\boldsymbol{x}$ and exponent intervals $\boldsymbol{y} = [\underline{y},\overline{y}]$ with $\underline{y} < \overline{y}$ it holds $\mathbf{pow}3(\boldsymbol{x},\boldsymbol{y}) = \widetilde{\mathbf{pow}3}(\boldsymbol{x},\boldsymbol{y})$.

computing of **pow3**

$$\mathbf{pow3}_{\mathbb{F}} : \overline{\overline{\mathbb{IR}}} \times \overline{\overline{\mathbb{IR}}} \to \overline{\overline{\mathbb{IR}}}$$

$$(\boldsymbol{x}, \boldsymbol{y}) \mapsto$$

Tabelle: The value of $\mathbf{pow}2([\underline{x},\overline{x}],\{n\})$ with $\overline{x} < 0$ and $n \in \mathbb{Z}$

	n even	n odd
$0 \le n$	$[\overline{x}^n, \underline{x}^n]$	$[\underline{x}^n, \overline{x}^n]$
$n \le 0$	$[\underline{x}^n, \overline{x}^n]$	$[\overline{x}^n, \underline{x}^n]$