Integral approach for solving two-dimensional continuity equation *

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We deal with two-dimensional continuity equation equipped with suitable known coefficients, initial and boundary conditions. To solve this problem we describe integral approach based on exact equality of two spatial integrals over domains located at the neighboring temporal levels. The convergence order of investigated scheme depends on accuracy of integral approximation. Presented scheme has convergence of first order. Introduced approach allows to avoid Courant-Friedrichs-Lewy condition for time step. Thus it's more convenient for problems with huge velocity than traditional methods. Theoretical investigations are confirmed by numerical experiments.

1. The problem statement and the main theorem

Consider the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} + \frac{\partial (v\rho)}{\partial y} = 0 \tag{1}$$

in the domain $[0,T] \times D$ with $D = [0,1] \times [0,1]$. Functions $u(t,x,y)$, $v(t,x,y)$ are known in $[0,T] \times D$ and smooth enough. We suppose for simplicity that $\forall\, t \in [0,T]$ the following conditions are satisfied:

$$u(t,x,y)\big|_{y=0} = v(t,x,y)\big|_{y=0} = 0, \quad u(t,x,y)\big|_{y=1} = v(t,x,y)\big|_{y=1} = 0, \tag{2}$$

and

$$u(t,x,y)\big|_{x=1} \geq 0. \tag{3}$$

For unknown function $\rho(t,x,y)$ the following initial and boundary data are defined:

$$\rho(t,x,y)\big|_{t=0} = \rho_{\text{init}}(x,y) \quad \rho(t,x,y)\big|_{x=0} = \rho_{\text{left}}(t,y) \tag{4}$$

where ρ_{init}, ρ_{left} are known and smooth enough.

To introduce the numerical algorithm, firstly use two time layers $t_{k-1}, t_k \in [0,T]$ with time step $\tau = t_k - t_{k-1} > 0$. There are many numerical methods for solving this problem (see [1-5] and reference in them). But most of these methods have the tedious Courant-Friedrichs-Lewy restriction for time step. We suggest approach without this restriction.

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At temporal level t_k consider an arbitrary straight-edges quadrangle Ω with four nodes (t_k, x_n, y_n), $n = 1, 2, 3, 4$. For each quadrangle edge at segment $t \in [t_{k-1}, t_k]$ we construct the the "characteristic surface" S_n consisting of characteristics of equation (1) with beginning at this edge. All four surfaces S_n cross the plane $t = t_{k-1}$ and carve in it a curvilinear quadrangle

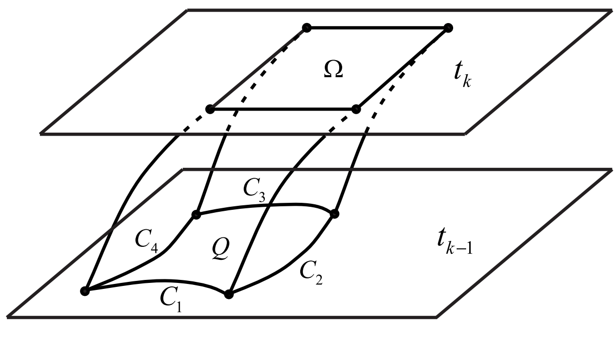

Рис. 1. Curvilinear quadrangle Q

Q with curved edges C_n (Fig. 1). If Ω is located near boundary of domain $[0, T] \times D$, surfaces S_n can cross the boundary. In this case we get additional curved lines which produce a

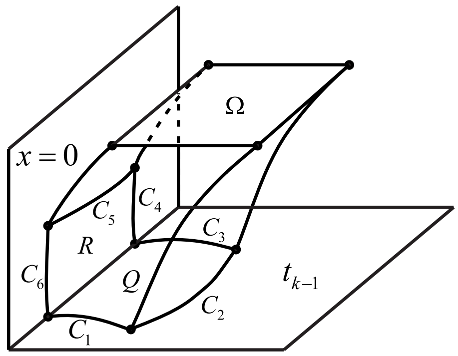

Рис. 2. Boundary quadrangle R

curvilinear quadrangle R (Fig. 2). Generally speaking, we may get a triangular or pentagonal domain R. Since it does not cause principal changes, we consider only the most common situation with quadrangular domain.

For Ω, Q, and R the following statement is valid.

Theorem 1. *For smooth solution of problem* (1) – (4) *we have the equality*

$$\int_{\Omega} \rho(t_k, x, y)\, dx\, dy = \int_{Q} \rho(t_{k-1}, x, y)\, dQ + I(R) \quad \text{where}$$

$$I(R) = \begin{cases} \displaystyle\int_R \rho\, u\, dR, & \text{if } R \neq \emptyset \text{ at the plane } x = 0, \\ 0, & \text{if } R = \emptyset. \end{cases}$$

2. Simple semi-discrete approximation

Now construct mesh D_h in plane Oxy. For simplification we use the uniform mesh D_h with nodes $(x_i, y_j):\ x_i = ih,\ y_j = jh,\ i, j = 0,\ 1,\ 2,\ \ldots, N$, and meshsize $h = 1/N$. Then we divide time segment $[0, T]$ by $M+1$ points $t_k = k\tau,\ \ k = 0,\ 1,\ 2,\ \ldots,\ M$, with time step

$\tau = T/M$ where $M \geq 1$. Suppose that the (approximate) discrete solution $\rho^h(t_{k-1}, x, y)$ at time level t_{k-1} is already known and we need to construct the approximate solution at time level t_k. For this purpose we need to take square $\Omega_{i,j}$ with four nodes $(x_i \pm h/2, y_j \pm h/2)$ and apply the Theorem 1. Thus we get

$$\int\limits_{x_{i-1/2}}^{x_{i+1/2}} \int\limits_{y_{j-1/2}}^{y_{j+1/2}} \rho(t_k, x, y)\, dy\, dx = \int_{Q_{i,j}^{k-1}} \rho(t_{k-1}, x, y)\, dQ + I\big(R_{i,j}^{k-1}\big) \quad \text{where} \tag{5}$$

$$I\big(R_{i,j}^{k-1}\big) = \begin{cases} \displaystyle\int_{R_{i,j}^{k-1}} \rho\, u\, dR, & \text{if } R_{i,j}^{k-1} \neq \emptyset \text{ at the plane } x = 0, \\ 0, & \text{if } R_{i,j}^{k-1} = \emptyset. \end{cases} \tag{6}$$

To compute integrals in (5) we use bilinear interpolation [6] introduced by basic functions $\psi_{p,q}(x, y) = \varphi_p(x)\,\varphi_q(y)$ where

$$\varphi_p(x) = \begin{cases} (x - x_{p-1})/h, & \text{if } x \in (x_{p-1}, x_p]\,, \\ (x_{p+1} - x)/h, & \text{if } x \in (x_p, x_{p+1}]\,, \\ 0 & \text{else.} \end{cases}$$

Thus $\forall\, r = 0, ..., M$ in each square $[x_i,\, x_{i+1}) \times [y_j,\, y_{j+1})$ we define interpolation of discrete function $\rho^h(t, x, y)$ by formula

$$\rho^h_{\text{int}}(t_r, x, y) = \sum_{q=j}^{j+1} \sum_{p=i}^{i+1} \rho^h(t_r, x_p, y_q)\, \psi_{p,q}(x, y) \quad \forall\, (x, y) \in [x_i,\, x_{i+1}) \times [y_j,\, y_{j+1}).$$

To compute $I\big(R_{i,j}^{k-1}\big)$ from (6) we also use the bilinear interpolation $(\rho u) \approx (\rho u)^h_{\text{int}}$, where

$$(\rho u)^h_{\text{int}} = \sum_{q=j}^{j+1} \sum_{r=k-1}^{k} \rho(t_r, 0, y_q)\, u(t_r, 0, y_q)\, \psi_{t,q}(t, y) \quad \forall\, (t, y) \in [t_{k-1},\, t_k] \times [y_j,\, y_{j+1}).$$

So instead of exact equality (5) we get an approximate one with property

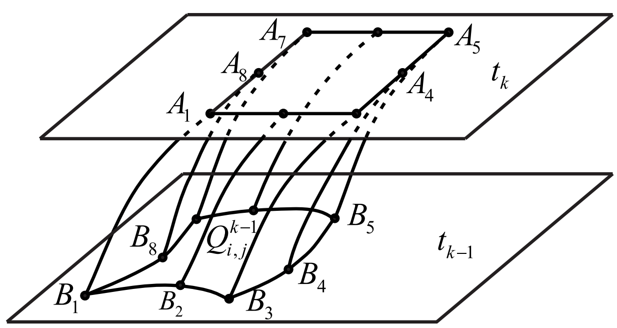

Рис. 3. Quadrangle approximation

$$\int\limits_{x_{i-1/2}}^{x_{i+1/2}} \int\limits_{y_{j-1/2}}^{y_{j+1/2}} \rho^h_{\text{int}}(t_k, x, y)\, dy\, dx \approx \int_{Q_{i,j}^{k-1}} \rho^h_{\text{int}}(t_{k-1}, x, y)\, dQ + I^h\big(R_{i,j}^{k-1}\big). \tag{7}$$

Left-hand side of (7) can be computed exactly

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \rho^h_{\text{int}}\left(t_k, x, y\right)\, dy\, dx = h^2 \rho^h\left(t_k, x_i, y_j\right). \tag{8}$$

To compute right-hand side of (7) we need to construct some approximation for domains $Q^{k-1}_{i,j}$ and $R^{k-1}_{i,j}$. Since both domains are quadrangles with curved sides, we demonstrate the approximation only for $Q^{k-1}_{i,j}$. Approximation of $R^{k-1}_{i,j}$ we can get by the same way. For this purpose consider four additional points $(x_i \pm h/2, y_j)$ and $(x_i, y_j \pm h/2)$ of square $\Omega_{i,j}$ at time level t_k and denote each of eight nodes by A_n, $n = 1, \ldots, 8$. From each A_n construct

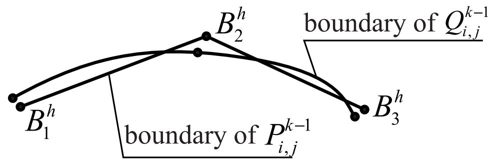

Рис. 4. Nodes approximation

corresponding characteristics to time level t_{k-1} which produces a point B_n (Fig. 3). To compute coordinates of point B_n approximately it's enough to solve the following system of ordinary differential equation:

$$\begin{cases} \dfrac{d\widetilde{x}_n}{dt} = u(t, \widetilde{x}_n, \widetilde{y}_n), \\ \\ \dfrac{d\widetilde{y}_n}{dt} = v(t, \widetilde{x}_n, \widetilde{y}_n), \end{cases} \qquad t \in [t_{k-1}, t_k],$$

with initial data

$$\widetilde{x}_n(t_k) = x_n, \quad \widetilde{y}_n(t_k) = y_n$$

by the Runge-Kutta method [7]. Thus we compute the approximate coordinates $B^h_n\big(\widetilde{x}_n(t_{k-1}), \widetilde{y}_n(t_{k-1})\big)$ of point B_n. All nodes B^h_n are producing the straight-edges polygon (octagon) $P^{k-1}_{i,j}$ which approximates domain $Q^{k-1}_{i,j}$ (Fig. 3, 4). By the same way we can construct polygon $L^{k-1}_{i,j}$ which approximates quadrangle $R^{k-1}_{i,j}$. Thus with the help of (7), (8) to compute the solution $\rho^h\left(t, x, y\right)$ at time level t_k we put

$$\rho^h\left(t_k, x_i, y_j\right) = \frac{1}{h^2} \int_{P^{k-1}_{i,j}} \rho^h_{\text{int}}\left(t_{k-1}, x, y\right)\, dP + \frac{1}{h^2} I^h\left(L^{k-1}_{i,j}\right), \quad \text{where} \tag{9}$$

$$I^h\big(L^{k-1}_{i,j}\big) = \begin{cases} \displaystyle\int_{L^{k-1}_{i,j}} (\rho u)^h_{\text{int}}\, dL, & \text{if } L^{k-1}_{i,j} \neq \varnothing \text{ at the plane } x = 0, \\ \\ 0, & \text{if } L^{k-1}_{i,j} = \varnothing. \end{cases}$$

To evaluate convergence we use the discrete analogue of norm in space $L_1\big([0,\, 1] \times [0,\, 1]\big)$:

$$\left\|\rho^h\right\|_{L^h_1} = \sum_{0 \le i,j \le N-1} \left|\rho^h\left(x_i, y_j\right)\right| h^2.$$

For numerical solution computed by (9) the following theorem is valid.

Theorem 2. *For sufficiently smooth solution* $\rho(t,x,y)$ *of problem* (1) – (4) *and discrete solution* $\rho^h(t,x,y)$ *computed by (9) we have the following estimate:*

$$\left\|\rho(t_k,\cdot)-\rho^h(t_k,\cdot)\right\|_{L_1^h} \le ckh^2 \quad \forall\ k=0,\,1,\ldots,M$$

with a constant c independent of k and h.

Corollary 1. *If the conditions of Theorem 2 are satisfied, for* $t_k=T$ *we have:*

$$\left\|\rho(T,\cdot)-\rho^h(T,\cdot)\right\|_{L_1^h} \le c_1T\frac{h^2}{\tau}.$$

Corollary 2. *If in addition to Theorem 2 we put* $\tau=c_2\,h$*, then*

$$\left\|\rho(T,\cdot)-\rho^h(T,\cdot)\right\|_{L_1^h} \le c_3Th.$$

Thus according to Corollary 2 we get the convergence of first order. Note that this approach does not require the validity of Courant-Friedrichs-Lewy condition [1, 2] for time step τ. Moreover, it is opposite in meaning: here the greater τ the better accuracy. Carried out numerical experiments completely confirm the theoretical investigation.

3. Conclusions

Up to now the numerical modeling of real physical process with large velocity values is quite difficult problem. One of its cause consists in necessity of too much time step for numerical computations. Considered numerical algorithm for the continuity equation allows to avoid Courant-Friedrichs-Lewy condition for time step. Thus we can provide numerical modeling with greater time step and smaller computational time. The validity of considered algorithm is confirmed by numerical experiment.

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